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The Primitivity of Free Products of Associative Algebras

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INTRODUCTION

Let R_1 and R_2 be nontrivial associative algebras over the field F and let $R = R_1 * R_2$ be their free product. In this article we prove the following theorem. (Theorem 2):

The algebra $R = R_1 * R_2$ is primitive provided that the dimensions of R_1 and R_2 are not both equal to 2.

This result was motivated by Formanek's paper [2], where the primitivity of group algebras of free products of groups is proved.

Let e_i, $i \in I$; f_j, $j \in J$, be fixed bases of R_1 and R_2, respectively. We assume them to be chosen in such a way that $e_0 = f_0 = 1$. Let $B_1 = \{e_i\}$, $i \in I - \{0\}$; $B_2 = \{f_j\}$, $j \in J - \{0\}$. It is well known that the unity and all the monomials in $B_1 \cup B_2$ such that their neighboring factors come from different B_μ $(\mu = 1, 2)$ form a basis of the algebra $R = R_1 * R_2$.

We suppose in the sequel that $\dim R_1 \geqslant \dim R_2$ *and* $\dim R_1 > 2$; or, equivalently, that $\operatorname{card} B_1 \geqslant \operatorname{card} B_2$ and $\operatorname{card} B_1 > 1$.

It is easy to check that the elements $e_i f_j$, $e_i \in B_1$, $f_j \in B_2$ freely generate a free associative algebra (with unity) T_{12}.

THEOREM 1. *Let $R = R_1 * R_2$. If A is a nonzero ideal of R, $A \cap T_{12} \neq 0$.*

The proof of Theorem 2 is based on Theorem 1 and the method of Formanek, which was applied in [2] to construct in the group ring of free products of groups some proper left ideal M which is comaximal with any two sided ideal. The existence of such a left ideal implies the primitivity of the ring.

It is not difficult to check that this method of [2] together with Theorem 2.1 of [1] gives the following result.

Let R be the free product of two rings R_1 and R_2 over a common division subring K. Assume that the left dimensions of R_1 and R_2 over K are not both equal to 2. Then R is left primitive provided that one of the following conditions holds:

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1) card $K \leqslant \aleph_0$.

2) card $K \leqslant \dim_l R_1$ *or* card $K \leqslant \dim_l R_2$.

This suggests the following conjecture.

Conjecture. The free product of two rings R_1 and R_2 over a common division subring K is primitive if dim R_k ($k = 1, 2$) are not both equal to 2.

Finally, we remark that the following result follows easily from Theorem 2. (See Theorem 3).

Let $\mathscr{L}_1$ and $\mathscr{L}_2$ be Lie algebras over F such that $\dim \mathscr{L}_k > 1$ $(k = 1, 2)$ *and let $\mathscr{L} = \mathscr{L}_1 * \mathscr{L}_2$ be their free sum. Then $\mathscr{L}$ has a faithful irreducible representation.*

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A basic monomial will be called a B_1-monomial if its first element belongs to B_1; an arbitrary element $0 \neq r \in R$ is a B_1-element if all the basic monomials in its support are B_1-monomials.

A basic monomial has type B_1B_1 if its first and last elements belong to B_1; the monomials of types B_1B_2, B_2B_1, and B_2B_2 are similarly defined. The length of such monomials (which is defined in a natural way) is positive; the length of 1 is defined to be zero.

Finally, if $r \neq 0$ is an arbitrary element of R, then lts length $l(r)$ is the length of a monomial of maximal length in its support.

The following obvious fact will be used a few times in the proof of Theorem 1.

LEMMA 1. *Let π and τ be two basic monomials. Then $l(\pi\tau) = l(\pi) + l(\tau)$ iff simultaneously π ends with element of $B_1(B_2)$ and τ begins with element of $B_2(B_1)$.*

LEMMA 2. *Let π be a monomial of length $l \leqslant n$. Then $(e_1f_1)^{n+1}\pi$ is a linear combination of basic monomials which begin with e_1.*

Proof. We need only consider the case when π is a B_2-monomial, i.e., $\pi = f\pi_1$, where $f \in B_2$ and π_1 is either unity or a B_1-monomial. It follows easily that the element $(e_1f_1)\pi = e_1(f_1f\pi_1)$ is a linear combination of basic monomials which begin in e_1 and of B_2-monomials which have length $\leqslant l - 1$. The assertion now follows by induction on n.

Proof of Theorem 1. We can suppose that there exists $0 \neq b \in A$ such that all the monomials of maximal length in its support are B_1B_2-monomials.

Indeed, let $0 \neq a \in A$. If among the monomials of maximal length in the support of a there exists some monomial π which does not have type B_1B_2, we define

$$\begin{aligned} b &= e_1af_1, && \text{if} \quad \pi \quad \text{has type} \quad B_2B_1, \\ &= ea_1, && \text{if} \quad \pi \quad \text{has type} \quad B_2B_2, \\ &= af_1, && \text{if} \quad \pi \quad \text{has type} \quad B_1B_2. \end{aligned}$$

Let $l(b) = m$. It follows from Lemmas 1 and 2 that $0 \neq c = (e_1f_1)^{m+1}\, b$ is a B_1-element.[1]

If now $c \notin T_{12}$ then the support of c contains B_1B_1-monomials; let $\tau = e_{i_1}f_{j_1}e_{i_2}f_{j_2} \cdots e_{i_k}f_{j_k}e_{i_{k+1}}$ be some of them. Left-multiplying c by the element e_1f_1 if necessary, we can suppose that $k \geqslant 1$. Let

$$e_{i_{k+1}}e_1 = \alpha_0 + \sum_{i \neq 0} \alpha_i e_i . \tag{1}$$

$$f_{\gamma k}f_1 = \beta_0 + \sum_{\gamma \neq 0} \beta_\gamma f_\gamma \tag{1'}$$

We obtain from (1) the following congruence modulo the subspace T_{12}

$$\begin{aligned} (e_1f_1)\, \tau(e_1f_1) &= e_1f_1(e_{i_1}f_{i_1}e_{i_2}f_{i_2} \cdots e_{i_k})\, (f_{i_k}e_{i_{k+1}}e_1f_1) \\ &\equiv (\alpha_0\beta_0e_1f_1)\, e_{i_1}f_{i_1}e_{i_2}f_{i_2} \cdots e_{i_k} . \end{aligned} \tag{2}$$

Repeating this argument k times gives

$$(e_1f_1)^k\, \tau(e_1f_1)^k \equiv \lambda(e_1f_1)^k\, e_1 \,(\operatorname{mod} T_{12}), \qquad \lambda \in F. \tag{3}$$

As above, we now obtain

$$(e_1f_1)\,(e_1f_1)^k\, e_1(e_1f_1) \equiv \mu(e_1f_1)^k\, e_1 \quad (\operatorname{mod} T_{12}), \tag{4}$$

where $\mu \in F$ is defined from the decompositions of $e_1{}^2$ and $f_1{}^2$ and does not depend on k.

It follows from (3) and (4) that for any $l > k$

$$(e_1f_1)^l\, \tau(e_1f_1)^l \equiv \gamma(e_1f_1)^k\, e_1 \quad (\operatorname{mod} T_{12}), \qquad \gamma \in F. \tag{4'}$$

Moreover, since in (4) μ does not depend on k we now obtain from (4) and (4′) that for any $l > k$

$$(e_1f_1)^{l+1}\, \tau(e_1f_1)^{l+1} - \mu(e_1f_1)^l\, \tau(e_1f_1)^l \in T_{12} . \tag{5}$$

Take now any $n > \frac{1}{2}l(c)$, then

$$d = (e_1f_1)^{n+1}\, c(e_1f_1)^{n+1} - \mu(e_1f_1)^n\, c(e_1f_1)^n \in T_{12} .$$

[1] By right-multiplying c by $(e_1f_1)^{2m+1}$ we can obtain an element which ends with f_1; in its support, however, there can appear B_2B_2-monomials (e.g., the basic element f_1). We therefore complete the proof using other considerations.

It only remains to prove that $d \neq 0$. This follows, however, from the fact that the sum of all the monomials of maximal length in the support of d is a nonzero element $d_1 \in T_{12}$ and T_{12} is a free algebra; hence

$$(e_1 f_1)^{n+1} d_1 (e_1 f_1)^{n+1} - \mu (e_1 f_1)^n d_1 (e_1 f_1)^n \neq 0,$$

and the theorem is proven.

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Proof of Theorem 2. Let us remark first that without loss of generality we can suppose that R_k $(k = 1, 2)$ is an algebra with unity. In fact, adjoining a unity if necessary and denoting the new algebra by $\tilde{R}_k$ $(k = 1, 2)$ we see that R is an ideal in the algebra $\tilde{R} = \tilde{R}_1 * \tilde{R}_2$. The truth of the assertion would follow, therefore, from the primitivity of $\tilde{R}$ and the well-known fact that any ideal of a primitive ring is primitive (see [3, Chap. 11, Sect. 4).

If A is a nonzero ideal of R then there exists by Theorem 1 an element $0 \neq a \in A$ such that

$$a = \sum_{k=1}^{m} \lambda_k \pi_k ,$$

where π_k $(k = 1, 2, \ldots, m)$ are $B_1 B_2$-monomials. Let $l(a) = n$.

Consider the element

$$b = (e_1 f_1)^{n+1} a (e_2 f_1)^{n+1} = \sum_{k=1}^{m} \lambda_k \rho_k ,$$

where $\rho_k = (e_1 f_1)^{n+1} \pi_k (e_2 f_1)^{n+1}$. It can be checked easily that the elements $\rho_1, \rho_2, \ldots, \rho_m$ freely generate a free associative algebra.

Now let $b_1 = b$; $b_2 = [b_1, \rho_m] = \lambda_1 [\rho_1, \rho_m] + \lambda_2 [\rho_2, \rho_m] + \cdots + \lambda_{m-1} [\rho_{m-1}, \rho_m]$; $b_3 = [b_2, [\rho_{m-1}, \rho_m]]$;.... At the $(m - 1)$th step we conclude that the ideal A contains some nonzero commutator q in elements $\rho_1, \rho_2, \ldots, \rho_m$.

In every nontrivial ideal of R we take such a commutator and denote the set of these commutators by Q.

We now apply the method of [2] to construct a proper left ideal of R which is comaximal with any two sided ideal.

Consider two cases.

Case 1. Card B_1 is infinite. There exists a one-to-one correspondence $q \to e(q)$ between Q and some subset of B_1.

Let U be the left ideal generated by all the elements

$$u(q) = 1 + f_1 q e(q) + q e(q) f_1 , \qquad q \in Q.$$

Since all the monomials of maximal length in the support of the element xu, $x \in R$, end in either $e(q)$ or in $e(q)f_1$, it follows easily that the equation

$$1 = x_1u_1 + x_2u_2 + \cdots + x_su_s, \qquad x_r \in R, \qquad r = 1, 2, \ldots, s,$$

is impossible. This means that U is a proper ideal.

Since any two-sided ideal $A \subseteq R$ contains some commutator $q \in Q$ it follows that U is comaximal with A.

Case 2. Card B_1 is finite. In this case Q is countable: $Q = \{q_1, q_2, \ldots\}$; as above, it can be checked that the elements

$$v_n = 1 + q_n(e_1f_1)(e_2f_1)^n + f_1q_ne_1(f_1e_2)^n, \qquad n = 1, 2, \ldots,$$

generate a left ideal V which is comaximal with any two sided ideal of R.

The theorem is proved.

Remark. Theorem 2 remains true when the limitation on dim R_k $(k = 1, 2)$ is replaced by the condition: card B_k $(k = 1, 2)$ are not both equal to 1.

Conversely, when R_k $(k = 1, 2)$ is the group algebra of Z_2 and hence, card $B_k = 1$ $(k = 1, 2)$, it was remarked in [2, 4] that $R = R_1 * R_2$ is not primitive.

The following fact is undoubtedly known but I have been unable to find a reference.

LEMMA 3. *Let F be an arbitrary field, $\mathscr{L}_1$ and $\mathscr{L}_2$ be Lie algebras over F and $\mathscr{L} = \mathscr{L}_1 * \mathscr{L}_2$ be their free sum. Let $\mathscr{U}_k$ be the universal enveloping algebra of $\mathscr{L}_k$ $(k = 1, 2)$. Then the universal enveloping algebra $\mathscr{U}$ of $\mathscr{L}$ is isomorphic to the free product of $\mathscr{U}_1$ and $\mathscr{U}_2$.*

Proof. Let A be arbitrary associative algebra and ϕ_k be a homomorphism of $\mathscr{U}_k$ in A $(k = 1, 2)$. We need prove that there exists a unique homomorphism ϕ of $\mathscr{U}$ in A such that $\phi(\mathscr{U}_k) = \phi_k(\mathscr{U}_k)$ $(k = 1, 2)$.

The homomorphism ϕ_k defines uniquely a homomorphism $\tilde{\phi}_k$ of $\mathscr{L}_k$ $(k = 1, 2)$ into the Lie algebra A_L of A. The homomorphism $\tilde{\phi}_k$ (together with the embeddings of $\mathscr{L}_k$ into $\mathscr{L}$) define uniquely a homomorphism $\tilde{\phi}$ of $\mathscr{L}$ into A_L, such that $\tilde{\phi}(\mathscr{L}_k) = \tilde{\phi}_k(\mathscr{L}_k)$ $(k = 1, 2)$. Finally, we can find a homomorphism ϕ of $\mathscr{U}$ in A_L such that it coincides with $\tilde{\phi}$ on $\mathscr{L}$. Hence, $\phi(\mathscr{L}_k) = \tilde{\phi}_k(\mathscr{L}_k)$ and this implies $\phi(\mathscr{U}_k) = \phi_k(\mathscr{U}_k)$ $(k = 1, 2)$.

The lemma is proved.

THEOREM 3. *Let F be an arbitrary field, and $\mathscr{L}_1$, $\mathscr{L}_2$ be Lie algebras over F such that* dim $\mathscr{L}_k > 1$ $(k = 1, 2)$ *and $\mathscr{L} = \mathscr{L}_1 * \mathscr{L}_2$ be their free sum. Then $\mathscr{L}$ has a faithful irreducible representation.*

Proof. Lemma 3 implies that the universal associative enveloping algebra $\mathscr{U}$ of $\mathscr{L}$ is the free product of the universal enveloping $\mathscr{U}_k$ of Lie algebras $\mathscr{L}_k$

$(k = 1, 2)$. We see also that $\dim \mathscr{U}_k > 2$ $(k = 1, 2)$ and Theorem 2 implies that there exists a faithful irreducible representation of $\mathscr{U}$ which means the existence of a faithful irreducible representation of $\mathscr{L}$.

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